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Generalizations of Arakawa's Jacobian

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A simple method yields discrete Jacobians that obey analogues of the differential properties needed to conserve energy and enstrophy in 2-dimensional flow. The method is actually independent of the type of discretization and thus applies to an arbitrary representation in gridpoints, finite elements, or spectral modes, or to any mixture of the three. We illustrate the method by deriving simple energy- and enstrophy-conserving Jacobians for an irregular triangular mesh in a closed domain, and for a mixed gridpoint-and-mode representation in a semi-infinite channel. © 1989 Academic Press, Inc.

1. INTRODUCTION

The motion of a 2-dimensional, inviscid incompressible fluid in a simply-connected region $\mathscr{D}$ of the xy-plane bounded by curve Γ, is governed by

$$\partial\zeta/\partial t = J(\zeta, \psi) \qquad \text{within } \mathscr{D} \text{ and on } \Gamma, \tag{1.1}$$

and the boundary condition

$$\psi = 0 \qquad \text{on } \Gamma. \tag{1.2}$$

Here, ψ is the streamfunction of the flow,

$$\zeta = \nabla^2\psi \tag{1.3}$$

is the vorticity, and

$$J(A, B) \equiv \partial A/\partial x\, \partial B/\partial y - \partial B/\partial x\, \partial A/\partial y \tag{1.4}$$

is the Jacobian operator, defined for any two functions $A(x, y)$ and $B(x, y)$. If either A or B is zero on Γ, then

$$\iint d\mathbf{x}\, AJ(A, B) = = \iint d\mathbf{x}\, BJ(A, B) = 0 \tag{1.5}$$

and

$$\iint d\mathbf{x}\, J(A, B) = 0, \tag{1.6}$$

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where the integration is over $\mathscr{D}$. It follows easily from the properties (1.5) and (1.6) that the motion governed by (1.1)–(1.3) conserves the energy

$$\frac{1}{2}\iint d\mathbf{x}\, \nabla\psi \cdot \nabla\psi, \tag{1.7}$$

the enstrophy

$$\frac{1}{2}\iint d\mathbf{x}\, \zeta^2, \tag{1.8}$$

and the mean vorticity

$$\iint d\mathbf{x}\, \zeta. \tag{1.9}$$

Arakawa [1] discovered a finite-difference analogue of the Jacobian operator (1.4) that obeys finite-difference analogues of (1.5)–(1.6). Because of these properties, numerical solutions of (1.1) using Arakawa's Jacobian conserve difference analogues of (1.7)–(1.9). (Here, and throughout this paper, we disregard errors resulting from difference approximations to time derivatives. Experience shows that these errors can always be reduced to acceptable levels.)

The conservation of energy and enstrophy guarantees numerical stability and prevents the spurious transfer of large amounts of energy to small lengthscales of the fluid motion. This spurious transfer is a characteristic property of numerical models that do not conserve enstrophy. Arakawa's Jacobian has been widely used for the numerical solution of (1.1) and of other, more general equations governing fluid motion.

Fix [2] derived an entire class of finite-element Jacobians with properties analogous to (1.5)–(1.6). Jespersen [3] showed that Arakawa's Jacobian is also a member of this class.

In this paper, we derive an even more general class of discrete Jacobian operators that obey analogues of (1.5). Our method, outlined in Section 2, is very simple, and is actually *independent of the exact method of discretization.* It therefore applies to an *arbitrary* representation in terms of gridpoints, finite elements, or spectral modes, or to any mixture of the three. We illustrate our method in Section 3 by deriving a simple energy- and enstrophy-conserving Jacobian for an irregular triangular mesh in a closed domain and for a mixed gridpoint-and-mode representation in a semi-infinite channel.

2. Method

The general equation

$$\mathscr{N}[\psi(x, y, t)] = 0 \qquad \text{in } \mathscr{D} \tag{2.1}$$

(where $\mathcal{N}$ is any operator) in the dependent variable $\psi(x, y, t)$ is equivalent to the requirement that

$$\iint d\mathbf{x}\, \alpha(x, y)\, \mathcal{N}[\psi(x, y, t)] = 0 \tag{2.2}$$

for *any* function $\alpha(x, y)$.

Consider the vorticity equation (1.1) in the form (2.2), namely

$$\iint d\mathbf{x}\, \alpha(\mathbf{x})\, \partial\zeta/\partial t = \iint d\mathbf{x}\, \alpha(\mathbf{x})\, J(\zeta, \psi). \tag{2.3}$$

Because of the boundary condition (1.2), the right-hand side of (2.3) can also be written as

$$\iint d\mathbf{x}\, \zeta J(\psi, \alpha) \tag{2.4}$$

or

$$\iint d\mathbf{x}\, \psi J(\alpha, \zeta). \tag{2.5}$$

Therefore (2.3) can be written in the general form

$$\iint d\mathbf{x}\, \alpha \partial\zeta/\partial t = \iint d\mathbf{x} \{ a\alpha J(\zeta, \psi) + b\zeta J(\psi, \alpha) + c\psi J(\alpha, \zeta) \}, \tag{2.6}$$

where a, b, c are any three constants that sum to unity,

$$a + b + c = 1. \tag{2.7}$$

The vorticity equation (1.1) is equivalent to the requirement that (2.6) hold for any choice of $\alpha(x, y)$. The conservation laws for energy and enstrophy correspond to the particular choices $\alpha = -\psi$ and $\alpha = \zeta$, respectively.

Now suppose that the integrals in (2.6) are replaced by sums over N gridpoints to obtain the finite-difference analogue

$$\begin{aligned} \sum_i \Omega_i \alpha_i \dot{\zeta}_i &= \sum_i \Omega_i \{ a\alpha_i J_i[\zeta_j, \psi_k] + b\zeta_i J_i[\psi_j, \alpha_k] + c\psi_i J_i[\alpha_j, \zeta_k] \} \\ &\equiv F(\alpha_1, ..., \alpha_n, \zeta_1, ..., \zeta_n, \psi_1, ..., \psi_n; a, b, c) \\ &\equiv F[\alpha_i, \zeta_j, \psi_k; a, b, c], \end{aligned} \tag{2.8}$$

where $(\alpha_i, \zeta_i, \psi_i)$ is (α, ζ, ψ) evaluated at the ith gridpoint, Ω_i is the area within $\mathcal{D}$ that is closest to the ith gridpoint, and

$$J_i[A_j, B_k] \equiv J_i(A_1, ..., A_n, B_1, ..., B_n) \tag{2.9}$$

is *any* finite-difference analogue of $J(A, B)$ at the ith gridpoint. Then

$$\Omega_i \dot{\zeta}_i = \partial F / \partial \alpha_i \tag{2.10}$$

is a finite-difference analogue of the exact vorticity equation (1.1). For a regular square grid of side $\hbar$, $\Omega_i = \hbar^2$ at every interior gridpoint.

The conservation properties of (2.10) are most easily established from (2.8). The discrete enstrophy

$$\tfrac{1}{2} \sum_i \Omega_i \zeta_i^2 \tag{2.11}$$

is conserved by (2.10) if the right-hand side of (2.8) vanishes when $\alpha_i = \zeta_i$ at all i. This follows by simple algebraic cancellations if $a = b$ and if the discrete Jacobian J_i has the antisymmetry property

$$J_i[A_j, B_k] = -J_i[B_j, A_k]. \tag{2.12}$$

Similarly, if (2.12) holds, and if $a = c$, then (2.8) vanishes when $\alpha_i = -\psi_i$, and therefore

$$-\sum_i \Omega_i \psi_i \dot{\zeta}_i = 0. \tag{2.13}$$

Equation (2.13) implies the conservation of a discrete analogue of the energy (1.7), provided that the discrete analogue of (1.3) takes a general form obtained below. We take up the question of mean vorticity conservation at the end of this section.

In summary, if the discrete Jacobian J_i is antisymmetric in its discrete arguments, then the discrete analogue

$$\Omega_i \dot{\zeta}_i = \partial / \partial \alpha_i F[\alpha_j, \zeta_k, \psi_l; \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}] \equiv \mathscr{J}_i[\zeta_j, \psi_k] \tag{2.14}$$

of the exact vorticity equation (1.1) obeys analogues of the Arakawa properties (1.5) and can conserve discrete analogues of the energy and enstrophy.

The discrete Jacobian $\mathscr{J}_i$ defined by (2.14) is the sought-for generalization of Arakawa's Jacobian. Note that the antisymmetry property (2.12) is the only restriction on the "initial" Jacobian J_i. In particular, J_i need *not* itself satisfy analogues of the Arakawa properties (1.5). The Jacobian $\mathscr{J}_i$ defined by (2.14) *does*, however, satisfy analogues of (1.5), as has just been proved.

The above procedure for obtaining $\mathscr{J}_i$ actually applies to finite-differences, finite elements, spectral truncations, or to any other general method of producing discrete approximations. To appreciate this, let (2.8) be replaced by the general discrete analogue

$$\sum_i \Omega_i \alpha_i \dot{\zeta}_i = \tfrac{1}{3} S[\alpha_i, \zeta_j, \psi_k] + \tfrac{1}{3} S[\zeta_i, \psi_j, \alpha_k] + \tfrac{1}{3} S[\psi_i, \alpha_j, \zeta_k], \tag{2.15}$$

where $S[A_i, B_j, C_k]$ is *any* discrete estimate of

$$\iint d\mathbf{x}\, AJ(B, C). \tag{2.16}$$

In the present context, the indices can represent gridpoints, finite-element nodes, spectral modes, or any combination thereof. (The weights Ω_i on the left-hand side of (2.15) have a different interpretation for the different methods of discretization. This point will be clarified by the examples of Section 3.) Again, the α_i-derivative of (2.15) is the discrete analogue of the exact vorticity equation (1.1). The right-hand side of (2.15) vanishes when $\alpha_i = \zeta_i$ (for all i), and when $\alpha_i = -\psi_i$, provided only that S is antisymmetric in its last two arguments, i.e.,

$$S[A_i, B_j, C_k] = -S[A_i, C_j, B_k]. \tag{2.17}$$

The requirement (2.17) is easy to satisfy. To see this, realize that any discrete approximation to (2.16) must take the general cubic form

$$S[A_i, B_j, C_k] = \sum_{ijk} D_{ijk} A_i B_j C_k, \tag{2.18}$$

where the coefficients D_{ijk} do *not necessarily* have the antisymmetry property $D_{ijk} = -D_{ikj}$ required by (2.17). However, the estimate

$$S[A_i, B_j, C_k] = \tfrac{1}{2} \sum_{ijk} (D_{ijk} - D_{ikj}) A_i B_j C_k, \tag{2.19}$$

obtained by replacing (2.18) by its antisymmetric part with respect to B and C, has the same order accuracy as (2.18), and is, moreover, exactly antisymmetric. Thus the "initial" Jacobian J_i and discrete estimate (2.18) for (2.16) can be selected on the basis of accuracy and convenience alone. The "refined" Jacobian $\mathcal{J}_i$ will automatically have the same order of accuracy *and* obey analogues of the Arakawa properties (1.5) at a cost of only a factor three in computation. Thus there are as many discrete Jacobians $\mathcal{J}_i$ with the Arakawa properties (1.5) as there are general nonconservative discrete Jacobians J_i.

The dynamics (2.14) based on the generalized Arakawa Jacobian $\mathcal{J}_i$ conserves an analogue of the energy (1.7) only if (2.13) is an exact time derivative. This depends on the discrete form of Eq. (1.3) relating the streamfunction and vorticity. Equation (1.3) is equivalent to the statement that

$$\iint d\mathbf{x}\, \beta\zeta = \iint d\mathbf{x}\, \beta\nabla^2\psi \tag{2.20}$$

for any function $\beta(x, y)$ that is zero on Γ. (The restriction on β is allowed because (1.3) is required only within $\mathscr{D}$.) An integration by parts brings (2.20) into the form

$$\iint d\mathbf{x}\, \beta\zeta = -\iint d\mathbf{x}\, \nabla\beta \cdot \nabla\psi. \tag{2.21}$$

We now discretize (2.21), being careful to discretize the left-hand side of (2.21) in the same way as the left-hand side of (2.8). The result is

$$\sum_i \Omega_i \beta_i \zeta_i = -\sum_i \Omega_i (\nabla\beta \cdot \nabla\psi)_i \tag{2.22}$$

where $(\nabla\beta \cdot \nabla\psi)_i$ is *any* discrete estimate of $\nabla\beta \cdot \nabla\psi$ at the ith gridpoint (node, or mode) that has the symmetry property

$$(\nabla\beta \cdot \nabla\psi)_i = (\nabla\psi \cdot \nabla\beta)_i. \tag{2.23}$$

The β_i-derivative of (2.22) is the discrete analogue of (1.3). Now, differentiate (2.22) with respect to time and set $\beta_i = \psi_i$ for all i. By (2.23), the resulting expression is an exact time derivative, and it then follows from (2.13) that the discrete energy

$$\tfrac{1}{2} \sum_i \Omega_i (\nabla\psi \cdot \nabla\psi)_i \tag{2.24}$$

is conserved.

The discrete dynamics also conserves the analogue

$$\sum_i \Omega_i (1)_i \zeta_i \tag{2.25}$$

of the mean vorticity (1.9) provided that (2.8) vanishes when $\alpha_i = (1)_i$ for all i. Here, $(1)_i$ is the discrete representation of unity, and $(1)_i = 1$ for gridpoints or nodes but not necessarily for spectral modes. Unfortunately, the general algorithm outlined above does not guarantee the conservation of (2.25). However, this is unsurprising because it is well known that spectral approximations, which are included in our method, do not generally conserve mean vorticity in bounded domains.

We can summarize broadly as follows. The steps required to establish energy and enstrophy conservation from the exact dynamical equations fall into two general categories: integrations by parts and simple algebraic cancellations. However, only the latter are easily carried over to discrete representations. The essence of our method is, loosely speaking, to perform the parts integrations *before* discretization. The remaining steps, which involve only multiplications and additions, transfer easily from the continuous to the discrete representations.

In the remainder of this paper, we use the method given above to derive discrete Jacobians that conserve analogues of the energy and enstrophy. The examples of Section 3, which offer a concrete illustration of our method, include a new conservative Jacobian that mixes gridpoints in one direction with spectral modes in the other.

3. Examples

A. *Arakawa's Jacobian*

As a first example, we consider 2-dimensional flow in a rectangular region of the (x, y) plane and obtain a familiar result. Let the flow domain be covered by a regular square grid with grid-spacing h, and let ψ_i and ζ_i be the streamfunction and vorticity at the ith gridpoint. Let the discrete dynamics be the α_i derivatives of (2.15) in the form

$$\sum_{\text{gridboxes}} \tfrac{1}{4}h^2(\alpha_1\dot{\zeta}_1+\alpha_2\dot{\zeta}_2+\alpha_3\dot{\zeta}_3+\alpha_4\dot{\zeta}_4)$$
$$=\tfrac{1}{3}\{S[\alpha_i,\zeta_j,\psi_k]+S[\zeta_i,\psi_j,\alpha_k]+S[\psi_i,\alpha_j,\zeta_k]\} \tag{3.1}$$

with

$$S[A_i,B_j,C_k]=\sum_{\text{gridboxes}} \tfrac{1}{4}(A_1+A_2+A_3+A_4)$$
$$\times\tfrac{1}{2}\{(B_2-B_4)(C_3-C_1)-(C_2-C_4)(B_3-B_1)\} \tag{3.2}$$

as the discrete approximation to

$$\iint d\mathbf{x}\, AJ(B,C). \tag{3.3}$$

The right-hand side of (3.1) is a sum over all *gridboxes* in the flow domain, and, *inside this sum only*, the subscripts refer to the representative gridbox shown in Fig. 1a. A summation over gridboxes is preferable to a summation over gridpoints, because the former does not require modification near the boundaries. The discretization (3.2) of (3.3) links every gridpoint to the smallest possible number of adjacent gridpoints and is, in that sense, the simplest choice possible. It can easily be checked that $\mathscr{J}_i\equiv\dot{\zeta}_i$ defined by (3.1), (3.2) is identical at interior gridpoints to the Jacobian discovered by Arakawa [1, Eq. (45)]. At the boundary points (Fig. 1b), where $\psi=0$, the formulae (3.1), (3.2) yield

$$\dot{\zeta}_0=1/(6h^2)\{\psi_2(\zeta_1-\zeta_3)+\psi_3(\zeta_1+\zeta_2-\zeta_4-\zeta_5)+\psi_4(\zeta_3-\zeta_5)\} \tag{3.4}$$

and at the corner points (Fig. 1c),

$$\dot{\zeta}_0=1/(3h^2)\{\psi_2(\zeta_1-\zeta_3)\}. \tag{3.5}$$

The terms in (3.4), (3.5) can be grouped into averages of finite-difference approximations to $J(\zeta,\psi)$ in several ways.

B. *Irregular Triangular Mesh*

Next consider 2-dimensional flow in an *arbitrary* bounded region $\mathscr{D}$ of the plane. Let $\mathscr{D}$ be covered by a triangular mesh as shown in Fig. 2. The elementary triangles,

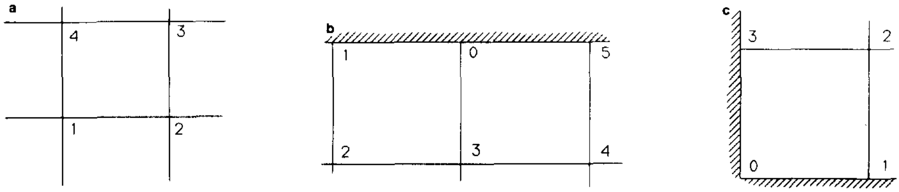

FIG. 1. Representative gridboxes (a) in the interior; (b) near the boundary; and (c) at a corner.

which have various shapes and sizes, fit closely against the boundary curve Γ. The discrete dependent variables are the values of α, ψ, and ζ at the mesh-points. Replace (2.6) by

$$\sum_{\text{triangles}} \tfrac{1}{3}\Omega_{\text{tr}}(\alpha_1\dot{\zeta}_1+\alpha_2\dot{\zeta}_2+\alpha_3\dot{\zeta}_3)$$
$$=\tfrac{1}{3}S[\alpha_i,\zeta_j,\psi_k]+\tfrac{1}{3}S[\zeta_i,\psi_j,\alpha_k]+\tfrac{1}{3}S[\psi_i,\alpha_j,\zeta_k], \tag{3.6}$$

where the summation is over all triangles in $\mathscr{D}$ and, *within each triangle of the sum*, the integers 1, 2, 3 denote the vertices, numbered counterclockwise, as shown for a representative triangle in Fig. 3a. Ω_{tr} is the triangle area. We choose

$$S[A_i,B_j,C_k]=\sum_{\text{triangles}} \tfrac{1}{3}\Omega_{\text{tr}}(A_1+A_2+A_3)[J(B,C)]_{\text{tr}} \tag{3.7}$$

as the discrete approximation to (3.3), where

$$[J(B,C)]_{\text{tr}}=\tfrac{1}{2}\Omega_{\text{tr}}^{-1}(B_2C_3-B_1C_3-B_2C_1-B_3C_2+B_1C_2+B_3C_1) \tag{3.8}$$

is a discrete approximation to $J(B, C)$ on the triangle. The estimate (3.8) is exact if B and C depend linearly on $\mathbf{x}$ within the triangle. Note that S given by (3.7) satisfies the antisymmetry property (2.17). Thus the Jacobian obtained by differentiating (3.6) with respect to each α_i has analogues of the Arakawa properties (1.5).

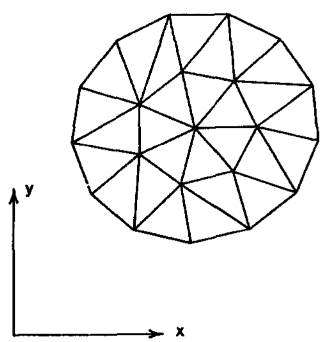

FIG. 2. A closed domain covered by a triangular mesh.

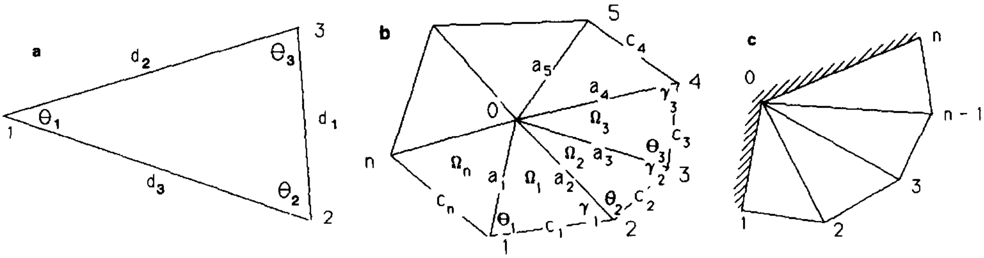

FIG. 3. A representative (a) triangle; (b) interior meshpoint; and (c) boundary meshpoint.

Now let subscript zero denote a representative interior meshpoint, and let 1, 2, 3, ..., n denote the n surrounding points, numbered counter-clockwise. Refer to Fig. 3b. Then straightforward manipulation on (3.6)–(3.8) yields

$$\left(\sum_{i=1,n} \Omega_i\right)\dot{\zeta}_0 = \frac{1}{2} \sum_{i=1,n} \zeta_i(\psi_{i+1} - \psi_{i-1}), \tag{3.9}$$

where the first sum is the total area of all the triangles with meshpoint zero as a vertex, and $\psi_0 \equiv \psi_n$, $\psi_{n+1} \equiv \psi_1$. The right-hand side of (3.9) is a discrete approximation to the integrated flux of vorticity into the region shown in Fig. 3b. At boundary points (Fig. 3c), the sum in (3.9) is replaced by

$$\zeta_1\psi_2 + \sum_{i=2,n-1} \zeta_i(\psi_{i+1} - \psi_{i-1}) - \zeta_n\psi_{n-1}. \tag{3.10}$$

The formula (3.9) was suggested by Williamson [4], who also noted its connection with Arakawa's scheme.

To obtain an analogue of Eq. (1.3) relating the vorticity and streamfunction, we replace (2.21) by the approximation

$$\sum_{\text{triangles}} \tfrac{1}{3}\Omega_{\text{tr}}(\beta_1\zeta_1 + \beta_2\zeta_2 + \beta_3\zeta_3) = -\sum_{\text{triangles}} \Omega_{\text{tr}}(\nabla\beta\cdot\nabla\psi)_{\text{tr}}, \tag{3.11}$$

where the integer subscripts again refer to Fig. 3a, and

$$(\nabla\beta\cdot\nabla\psi)_{\text{tr}} \equiv (4\Omega_{\text{tr}})^{-2}\{\beta_1\psi_1 d_1^2 - \beta_1\psi_2 d_1 d_2 \cos\theta_3 - \beta_1\psi_3 d_1 d_3 \cos\theta_2 + \cdots\} \tag{3.12}$$

(plus six additional terms whose form follows by symmetry) is a discrete approximation to $\nabla\beta\cdot\nabla\psi$ on the triangle. Again, the estimate (3.12) is exact if both β and ψ depend linearly on **x**. That is, (3.12) is derived by assuming that, within each triangle,

$$\beta(x, y) = A + Bx + Cy \qquad \text{and} \qquad \psi(x, y) = D + Ex + Fy \tag{3.13}$$

with the six constants A, B, C, D, E, F determined by the three nodal values of β and ψ. Now let subscript zero again correspond to the center point of Fig. 3b (a representative interior meshpoint), and take the β_0-derivative of (3.11). The result is

$$\left(\sum_{i=1,n} \Omega_i\right)\zeta_0 = -\tfrac{3}{4}\left(\sum_{i=1,n} c_i^2 \Omega_i^{-1}\right)\psi_0 + \tfrac{3}{4}\sum_{i=1,n} \Omega_i^{-1}(c_i a_{i+1}\cos\gamma_i \psi_i + c_i a_i \cos\theta_i \psi_{i+1}), \qquad (3.14)$$

where the summations are over the n triangles surrounding the central meshpoint, and the lengths, areas, and angles are defined in Fig. 3b. Equation (3.14) determines the streamfunction from the vorticity at all interior meshpoints.

By the general theory of Section 2, the discrete dynamics (3.9), (3.14) conserves the energy

$$\tfrac{1}{2}\sum_{\text{triangles}} \Omega_{\text{tr}}(\nabla\psi\cdot\nabla\psi)_{\text{tr}} \qquad (3.15)$$

and the enstrophy

$$\tfrac{1}{2}\sum_{\text{triangles}} \Omega_{\text{tr}}\tfrac{1}{3}(\zeta_1^2+\zeta_2^2+\zeta_3^2). \qquad (3.16)$$

For the simple case in which Fig. 3b is a hexagon composed of equilateral triangles of side d, we have

$$c_i = a_i = d, \qquad \Omega_i = \sqrt{3}/4\, d^2, \qquad \gamma_i = \theta_i = \pi/3 \qquad (3.17)$$

and the dynamics (3.9), (3.14) takes the simple forms

$$\dot{\zeta}_0 = 1/(3\sqrt{3}\, d^2)\sum_{i=1,6} \zeta_i(\psi_{i+1}-\psi_{i-1}) \qquad (3.18)$$

and

$$\zeta_0 = 2/(3d^2)\sum_{i=1,6}(\psi_i - \psi_0) \qquad (3.19)$$

previously given by Masuda [5] and Sadourny, Arakawa, and Mintz [6].

The triangular mesh is an important example because an arbitrary *curved* surface in 3-dimensional space can be replaced by a surface consisting of piecewise planar triangles. If, for example, $\mathscr{D}$ is a sphere, then the approximating surface would resemble a "geodesic dome." *Within* each triangle, the surface geometry is flat, and (3.9), (3.14) apply without change. Moreover, since (3.9), (3.14) are coordinate-free (that is, they make no reference to a global system of coordinates) there can never be problems with coordinate-system singularities like the convergence of meridians at the poles of a sphere.

C. *Spectral and Finite Element Methods*

It is trivial to show that the general method of Section 2 includes spectral and finite-element approximations. We simply discretize all the exact equations by replacing α, ζ, and ψ by the finite sums

$$\alpha = \sum_{i=1,N} \alpha_i(t)\varphi_i(\mathbf{x}), \qquad \text{etc.,} \tag{3.20}$$

where $\varphi_i(\mathbf{x})$ are othogonal functions in the case of spectral approximation and shape functions in the case of finite elements. The discrete approximation

$$\sum_{i,j,k} A_i B_j C_k \iint d\mathbf{x}\, \varphi_i J(\varphi_j, \varphi_k) \tag{3.21}$$

to (3.3) is obviously antisymmetric in B and C. However, the integrals in (3.21) can be awkward to compute, and, since, as shown in Section 2, the conservation of energy and enstrophy depends only on algebraic cancellations, there seems little reason not to use the generally simpler finite-differences. On the other hand, spectral approximations have the highest order accuracy, and, as shown by Fix [2], finite-element approximations can be arranged to conserve an analogue of the mean vorticity (1.9). Of course, the same discrete equations often (perhaps always) correspond to both a finite-difference and a finite-element approximation.

D. *A Hybrid Spectral and Finite-Difference Jacobian*

Finally, we consider the dynamics (1.1), (1.3) in an infinite channel with walls at $y=0$ and $y=W$. The boundary conditions are that the flow be periodic in the x-direction, and that ψ be zero at $y=0$ and equal to a prescribed constant at $y=W$. The discrete representation consists of N gridpoints with spacing Δy across the channel, and $2L+1$ Fourier modes along the channel, including an x-independent "mean flow." Thus,

$$\psi(x, r\Delta y, t) = \sum_{m=-L,L} \psi_m^r(x, t), \qquad r = 1 \quad \text{to } N, \tag{3.22}$$

where

$$\psi_m^r(x, t) \equiv \psi_{mr}(t) \exp\{ik_m x\} \tag{3.23}$$

and similarly for ζ and α, with $\psi_{0N} = \text{constant}$, $\psi_{mN} = 0$ $(m \neq 0)$, and $\psi_{m0} = 0$. We write (2.15) in the form

$$R[\alpha_{ij}, \zeta_{kl}] = \tfrac{1}{3}S[\alpha_{ij}, \zeta_{kl}, \psi_{mn}] + \tfrac{1}{3}S[\zeta_{ij}, \psi_{kl}, \alpha_{mn}] + \tfrac{1}{3}S[\psi_{ij}, \alpha_{kl}, \zeta_{mn}], \tag{3.24}$$

where

$$R[\alpha_{ij}, \dot{\zeta}_{kl}] = \int dx \sum_{r=0,N-1} \Delta y \sum_m \sum_n \tfrac{1}{2}\{\alpha_m^r \dot{\zeta}_n^r + \alpha_m^{r+1} \dot{\zeta}_n^{r+1}\}, \tag{3.25}$$

$$S[A_{ij}, B_{kl}, C_{mn}] = T[A_{ij}, B_{kl}, C_{mn}] - T[A_{ij}, C_{kl}, B_{mn}], \tag{3.26}$$

and

$$T[A_{ij}, B_{kl}, C_{mn}] = \int dx \sum_{r=0,N-1} \Delta y \left\{ \left[\sum_m \sum_j ik_j \tfrac{1}{2}(A_m^r B_j^r + A_m^{r+1} B_j^{r+1}) \right] \times \left[\sum_n (C_n^{r+1} - C_n^r)/\Delta y \right] \right\}. \tag{3.27}$$

Here, (3.26) is perhaps the simplest discrete approximation to (3.3), with (3.27) a discrete approximation to

$$A\, \partial B/\partial x\, \partial C/\partial y. \tag{3.28}$$

The discrete analogue of the vorticity equation (1.1) is obtained by taking the derivatives of (3.25) with respect to the $\alpha_{mr}(t)$. At interior gridpoints we obtain

$$\begin{aligned} \dot{\zeta}_{mr} = \mathscr{J}_{mr}[\zeta_{ij}, \psi_{kl}] \equiv i/(6\,\Delta y) \sum_{k_j + k_n = k_m} & \{2k_j \zeta_{jr}(\psi_{n,r+1} - \psi_{n,r-1}) \\ & + 2k_n \psi_{nr}(-\zeta_{j,r+1} + \zeta_{j,r-1}) \\ & + k_n \zeta_{jr}(\psi_{n,r+1} - \psi_{n,r-1}) - k_j \psi_{nr}(\zeta_{j,r+1} - \zeta_{j,r-1}) \\ & + k_j(\psi_{n,r+1}\zeta_{j,r+1} - \psi_{n,r-1}\zeta_{j,r-1}) \\ & - k_n(\psi_{n,r+1}\zeta_{j,r+1} - \psi_{n,r-1}\zeta_{j,r-1})\}. \end{aligned} \tag{3.29}$$

The Jacobian defined by (3.29) corresponds to the exact Jacobian $J(\zeta, \psi)$ in the form

$$J(\zeta, \psi) = \tfrac{1}{3}\{2\zeta_x \psi_y - 2\psi_x \zeta_y + \zeta(\psi_y)_x - \psi(\zeta_y)_x + (\psi\zeta_x - \zeta\psi_x)_y\} \tag{3.30}$$

with y-derivatives replaced by centered differences.

Energy conservation demands a consistent discrete analogue of (1.3). For this we write (2.22) as

$$R[\beta_{ij}, \zeta_{kl}] = -\int dx \sum_{r=0,N-1} \Delta y \sum_m \sum_n \{-k_m k_n \tfrac{1}{2}(\beta_m^r \psi_n^r + \beta_m^{r+1} \psi_n^{r+1}) + (\beta_m^{r+1} - \beta_m^r)/\Delta y\; (\psi_n^{r+1} - \psi_n^r)/\Delta y\} \tag{3.31}$$

and note that the left-hand side of (3.31) matches that of (3.24). The β_{mr}-derivative of (3.31) yields the analogue

$$\zeta_{mr} = -k_m^2 \psi_{mr} + \{\psi_{m,r+1} - 2\psi_{mr} + \psi_{m,r-1}\}/(\Delta y)^2 \tag{3.32}$$

of (1.3) at interior gridpoints. It can be verified that the discrete dynamics (3.29), (3.32) conserves discrete analogues of the energy and enstrophy. Since (3.24), (3.31) are probably the simplest discrete analogues of (2.15), (2.23) for the chosen hybrid representation, this discrete dynamics is probably the simplest conserving scheme possible.

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